Cutting planes by tilting

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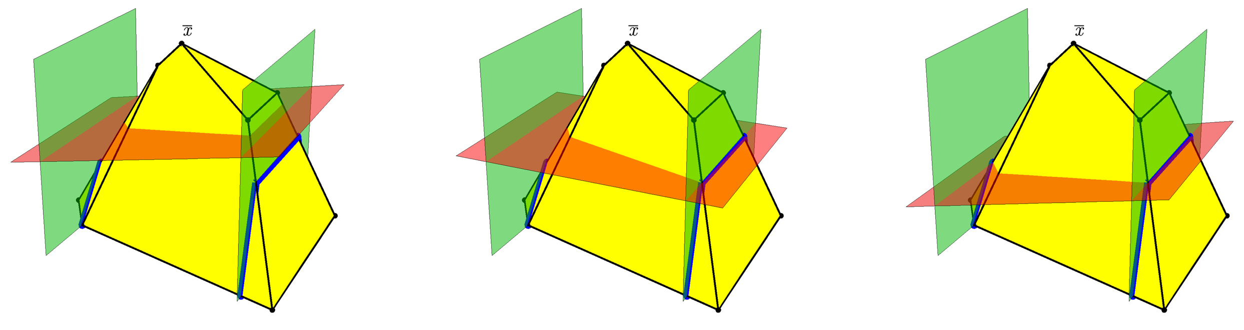
THEORY

Input: (LP): $\min\{cx : x \in P\}$ where $P = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\}$ and A is a rational $m \times n$ matrix.
(MILP): $\min\{cx : x \in P_I\}$ where $P_I = \{x \in P : x_j \in \mathbb{Z},\ j \in I\}$ for a subset $I \subseteq [n] = \{1, \dots, n\}$.
$\bar{x}$: An optimal solution to (LP).

Goal: Tighten relaxation of P_I via valid inequalities cutting off $\bar{x}$.

Throughout, S will denote a split set $\{x : \pi_0 \leq \pi x \leq \pi_0 + 1\}$ satisfying $P_I \cap \operatorname{int} S = \emptyset$ and $\bar{x} \in \operatorname{int} S$.

Let $F^0 = \{x : \pi x = \pi_0\}$ and $F^1 = \{x : \pi x = \pi_0 + 1\}$.



Theorem.
Let β^0 and β^1 be given such that, for $q \in \{0, 1\}$, $hx \geq \beta^q$ is valid for $P \cap F^q$.

Then the inequality $\alpha x \geq \beta$ as defined below is valid for $\operatorname{conv}(P \setminus \operatorname{int} S)$ if either the inequality cuts a vertex of $P \cap \operatorname{int} S$ or there exists a point $p^q \in P \cap F^q$ achieving $hp^q = \beta^q$ for $q = 0$ and $q = 1$.

$$\alpha_j = h_j + (\beta^0 - \beta^1)\pi_j, \quad j \in [n]$$
$$\beta = \beta^0 + (\beta^0 - \beta^1)\pi_0$$

COMPUTATION

Define $\sigma = \{j \in I : \bar{x}_j \notin \mathbb{Z}\}$ as the set of integer variables fractional at $\bar{x}$. Let $S_j = \{x : \lfloor \bar{x}_j \rfloor \leq x_j \leq \lceil \bar{x}_j \rceil\}$ be the *simple split* on x_j for $j \in \sigma$.

Experiments: Measured percentage of integrality gap closed by tilted cuts on 42 small (at most 500 rows and 500 columns) benchmark instances from MIPLIB, where the cuts were generated from all possible simple splits.

GMI cuts are used as benchmark. **Tilted cuts are added together with GMI cuts in results.**

How to tilt?

For $q \in \{0, 1\}$, set $\beta^q = \min\{hx : x \in P \cap F^q\}$.

Which inequalities to tilt?

(T0) Natural first choice: **objective vector**.
(T1) Other inequalities in input: **row and column bounds**.

	GMI	T0	T1
Gap closed	22.9	31.8	**35.2**
Time		0.1	**45.4**
# cuts / # GMI	1.0	1.0	**15.2**
# cuts / # bounds	0.1	0.1	**1.4**

Instance	GMI	T0	T1	T1*	T1*M	T$^+_{1K}$	T$^+_{10K}$	#GMI	#T1	#T1*	#T1*M	#T$^+_{1K}$	#T$^+_{10K}$
bell3a	45.1	61.5	64.0	64.0	61.6	64.6	64.6	27	523	125	36	458	458
misc01	0.0	0.0	0.0	0.0	0.0	**2.6**	2.6	12	410	36	11	170	170
mod013	4.4	7.4	7.4	7.4	5.7	**26.2**	26.2	5	5	5	1	57	57
p0282	3.7	40.3	85.7	81.3	**81.3**	**17.2**	**17.2**	26	510	64	39	1,000	1,190
prod1	0.0	40.9	64.3	63.3	**63.5**	**9.1**	**9.1**	53	**2,002**	196	130	1,000	**3,849**
rlp2	0.6	1.9	3.2	3.2	**3.1**	**2.7**	**2.7**	39	**1,982**	61	38	762	762
stein45*	7.1	27.7	27.7	27.7	23.7	27.8	31.4	45	46	45	**1**	1,000	**3,130**
timtab1	23.7	23.7	31.4	31.1	**31.8**	**26.4**	**34.2**	136	**3,431**	429	146	1,000	**10,000**

EXTENSIONS

Strength

Can we **mix** information from different splits?

Theorem. (Günlük and Pochet, 2001)
For $j \in \sigma$, let $\underline{\beta}_j = \min\{\beta^0, \beta^1\}$ and $\bar{\beta}_j = \max\{\beta^0, \beta^1\}$.
Let $\bar{\beta}_0 = \max\{\underline{\beta}_j : j \in \sigma\}$. Assume $\bar{\beta}_j \leq \bar{\beta}_{j'}$ if $j < j'$.
Then the following inequality is valid for P_I.

$$hx \geq \bar{\beta}_0 + \sum_{j \in \sigma} (\bar{\beta}_j - \bar{\beta}_{j-1}) x_j$$

This is tilting.

Speed

Too slow, too many cuts!

Idea: choose only a subset of the bounds to tilt.

(T1*) Only tilt bounds **tight** at $\min\{cx : x \in P \cap F^q\}$ for either $q = 0$ or $q = 1$.

Order of magnitude more efficient, but still doubling number of GMI cuts.

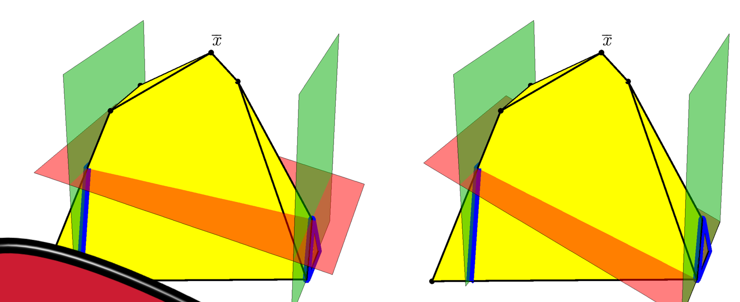
Both?

	GMI	T1	T1M	T1*	T1*M
Gap closed	22.9	35.2	35.4	**35.1**	**35.1**
Time		45.4	**45.8**	**3.1**	**3.1**
# cuts / # GMI	1.0	15.2	**4.4**	**2.0**	**0.8**
# cuts / # bounds	0.1	1.4	**0.3**	**0.2**	**0.1**

Other ways to use several splits at once?

(T$^+$) Tilt intersection cuts from $P \cap F^q$.

For validity, these are only generated from splits on binary variables.



Many, many cuts.

	T$^+_{1K}$	T$^+_{10K}$
Gap closed	35.8	**37.7**
Time	**0.6**	2.5
# cuts / # GMI	17.0	**26.5**
# cuts / # bounds	1.7	**2.8**

But strong and fast.

Conclusions

Results show we can use **tilting** to generate **strong** cuts **quickly**.

In fact, cuts can be generated for free while gathering **strong branching** information.

Many open questions left, both from the computational and theoretical sides.

Much more left to do!

REFERENCES

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3. Richard, J.-P. P. (2011). Lifting techniques for mixed integer programming. Wiley Encyclopedia of Operations Research and Management Science.